

ON THE EXISTENCE OF BOUND AND GROUND STATES FOR SOME COUPLED NONLINEAR SCHRÖDINGER–KORTEWEG-DE VRIES EQUATIONS

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Abstract. We demonstrate existence of positive bound and ground states for a system of coupled nonlinear Schrödinger–Korteweg-de Vries equations. More precisely, we prove there is a positive radially symmetric ground state if either the coupling coefficient $\beta > \Lambda$ (for an appropriate constant $\Lambda > 0$) or $\beta > 0$ with appropriate conditions on the other parameters of the problem. Concerning bound states, we prove there exists a positive radially symmetric bound state if either $0 < \beta$ is sufficiently small or $0 < \beta < \Lambda$ with some appropriate conditions on the parameters as for the ground states. That results give a classification of positive solutions as well as multiplicity of positive solutions. Furthermore, we consider a system with more general power nonlinearities, proving the above results, and also we study natural extended systems with more than two equations. Although the techniques we employed are variational, we look for critical points of an energy functional different from the classical one used in this kind of systems. Our approach improves many of the previous known results, as well as permit us to show new results not considered or studied before.

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1. INTRODUCTION

The aim of this work is to study a system of coupled nonlinear Schrödinger–Korteweg-de Vries (NLS-KdV for short) equations as follows,

$$\begin{cases} if_t + f_{xx} + |f|^2 f &= \beta f g \\ g_t + g_{xxx} + g g_x &= \frac{1}{2} \beta (|f|^2)_x \end{cases} \quad (1)$$

where $f = f(x, t) \in \mathbb{C}$ while $g = g(x, t) \in \mathbb{R}$, and $\beta > 0$ is the real coupling coefficient. System (1) appears in phenomena of interactions between short and long dispersive waves, arising in fluid mechanics, such as the interactions of capillary - gravity water waves. Indeed, f represents the short-wave, while g stands for the long-wave. See [2, 16, 21] and the references therein for more details.

We look for solitary “traveling-waves”, namely solutions to (1) of the form

$$(f(x, t), g(x, t)) = \left(e^{i\omega t} e^{i\frac{c}{2}x} u(x - ct), v(x - ct) \right) \quad \text{with } u, v \text{ real functions.} \quad (2)$$

Choosing $\lambda_1 = \omega + \frac{c^2}{4}$, $\lambda_2 = c$, we get that u, v solve the following problem

$$\begin{cases} -u'' + \lambda_1 u &= u^3 + \beta uv \\ -v'' + \lambda_2 v &= \frac{1}{2} v^2 + \frac{1}{2} \beta u^2. \end{cases} \quad (3)$$

This system has been previously studied by Dias et al. in [17]. Also a generalization of (3), see (48), has been previously analyzed by the same authors in [18] and by Albert and Bhattacharj in [3]. A comparison with our results will be done at the end of this introduction.

The main goal of this work is three fold. One is to give a classification of positive solutions of (3), proving:

-Existence of positive even ground states of (3) under the following hypotheses:

- the coupling coefficient $\beta > \Lambda > 0$ for an appropriate constant Λ ; see Theorem 6,
- $\beta > 0$ and $\lambda_2 \gg 1$; see Theorem 8.

-Existence of positive even bound states of (3) when:

- $0 < \beta \ll 1$; see Theorem 9, where we also give a bifurcation result,
- $0 < \beta < \Lambda$ and $\lambda_2 \gg 1$; see Theorem 8.

The coexistence of positive bound and ground states for $0 < \beta < \Lambda$ and λ_2 large is a great novelty and difference with the more studied systems of NLS equations in the last several years; see Remark 11-(ii).

The second goal is that we study a more general system than (3), with more general power nonlinearities given by (48), for which we show that previous results for (3) hold with similar conditions on the coefficients.

We also analyzed a particular case of λ_1, λ_2 in which there exists an explicit positive solution.

The last goal is to consider natural extensions of (3) to systems with more than two equations, as well as deal with extensions to the dimensional cases $n = 2, 3$, for which although (1) has no sense, the stationary system (3) makes sense and can be seen, for example, as the stationary system when one looks for standing wave solutions of the corresponding evolutionary system of NLS equations. For some of these extended problems we show similar results as described above on the existence of positive radially symmetric bound and ground state solutions. Other systems with at least two NLS equations and at least one KdV equation will be analyzed (in more detail) in a forthcoming paper.

Besides the previous achievements, it is relevant to point out that is the first time that our variational procedure (in part developed in [5, 6] for systems of coupled NLS equations) is employed to study coupled NLS-KdV equations in an appropriate way, see Remark 17-(ii). Even more, it seems to be better in many ways than the classical approach used before to study NLS-KdV systems as we will see. Note that our method could be exploited to study related problems.

It is worth pointing out that, for any $\beta \in \mathbb{R}$, System (3) has a unique *semi-trivial* positive solution $\mathbf{v}_2 = (0, V_2)$, where $V_2(x) = 3\lambda_2 \operatorname{sech}^2\left(\frac{\sqrt{\lambda_2}}{2}x\right)$ is the unique positive even solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2$ in $W^{1,2}(\mathbb{R})$; [25]. As might be expected, we look for different solutions from the preceding one. We are interested not in non-negative solutions but in positive ones, and therefore different from \mathbf{v}_2 .

As we announced above, a comparison with our results and the previous works [17, 18, 3] is in order. In [17], Dias et al. studied (3) in the particular case $\lambda_1 = \lambda_2$ and they proved the existence of non-negative bound state solutions when the coupling parameter $\beta > \frac{1}{2}$. Here, we have improved that result in three ways. First, we have considered λ_1 not necessarily equals λ_2 and we proved not only the existence of non-negative bound states but also positive even ground states for β greater than a constant $\Lambda > 0$ defined by (12), for which in the setting of [17], we have $\Lambda \leq \frac{1}{2}$. Secondly, we show the existence of positive even bound states when $0 < \beta \ll 1$, not studied in [17]. Thirdly, we also show that if λ_2 is sufficiently large, there exists a positive even ground state for every $\beta > 0$ (with λ_1 not necessarily equals λ_2) and a positive even bound state provided $0 < \beta < \Lambda$. In [18], among other results, Dias et al. studied System (48) with $2 < q < 5$, $p \in \{2, 3, 4\}$, $\mu_2 = p + 1$, and they established ([18, Theorem 4.1]) the existence of a non-trivial bound state solution for all $\beta > 0$ if $p = 3, 4$ and $\beta > 3$ if $p = 2$. Finally, in [3] Albert and Bhattacharai studied, among other topics, system (48) in a more general setting than in [18], precisely they considered $2 \leq q < 5$, $2 \leq p < 5$ with p a rational number with odd denominator; they proved the existence of a positive even bound state for each $\beta > 0$, improving the above cited result by [18]. In our manuscript we consider $2 \leq p < \infty$, $2 \leq q < \infty$ and we prove that there exists a positive even ground state of (48) if either $\beta > \Lambda$ or $\beta > 0$, $q > 2p - 2$ and λ_2 is large enough. Concerning bound states we show the existence of a positive even bound state of (48) if either $0 < \beta < \Lambda$, $q > 2p - 2$ and λ_2 is large enough or $0 < \beta \ll 1$, proving also in this last case a bifurcation result for the bound state we find.

By the discussion above of problem (48), we improve and extend some of the results by [18, 3]. Additionally, we establish some new results for (48), as the multiplicity one of coexistence of positive bound and ground states in some range of the parameters. Finally, we analyzed extended systems of (3) with more than two equations, that up to our knowledge, have not been considered previously in the literature. Also, we study the qualitative and quantitative properties of the explicit solutions of (3).

A preliminary announcement of some results in the present work appeared in [14].

The paper is organized as follows. In Section 2 we introduce the functional framework, notation and give some definition. Next, we define the Nehari Manifold in Section 3, proving some properties of it, we establish a useful measure lemma and show a result dealing with qualitative properties of the semi-trivial solution. Section 4.1 is divided into two subsections, the first one

contains the existence of ground states, and the second one deals with the existence of bound states. In Section 5 we study a system with more general power nonlinearities, proving similar results as in the previous one with the appropriate changes. Section 6 contains two subsections, the first one deals with an explicit solution, while the last one is devoted to study natural extensions to systems with more than two equations.

2. FUNCTIONAL SETTING AND NOTATION

Let E denotes the Sobolev space $W^{1,2}(\mathbb{R})$, that can be defined as the completion of $\mathcal{C}_0^1(\mathbb{R})$ endowed with the norm

$$\|u\| = \sqrt{(u | u)},$$

which comes from the scalar product

$$(u | w) = \int_{\mathbb{R}} (u'w' + uw) dx.$$

We will denote the following equivalent norms and scalar products in E ,

$$\|u\|_j = \|u\|_{\lambda_j} = \left(\int_{\mathbb{R}} (|u'|^2 + \lambda_j u^2) dx \right)^{\frac{1}{2}},$$

$$(u|v)_j = (u|v)_{\lambda_j} = \int_{\mathbb{R}} (u' \cdot v' + \lambda_j uv) dx; \quad j = 1, 2.$$

Let us define the product Sobolev space $\mathbb{E} = E \times E$. The elements in \mathbb{E} will be denoted by $\mathbf{u} = (u, v)$, and $\mathbf{0} = (0, 0)$. We will take

$$\|\mathbf{u}\| = \sqrt{\|u\|_1^2 + \|v\|_2^2}$$

as a norm in \mathbb{E} .

For $\mathbf{u} = (u, v) \in \mathbb{E}$, the notation $\mathbf{u} \geq \mathbf{0}$, resp. $\mathbf{u} > \mathbf{0}$, means that $u, v \geq 0$, resp. $u, v > 0$. We denote H as the space of even (radially symmetric) functions in E , and $\mathbb{H} = H \times H$.

We define the functionals

$$I_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{4} \int_{\mathbb{R}} u^4 dx, \quad I_2(v) = \frac{1}{2}\|v\|_2^2 - \frac{1}{6} \int_{\mathbb{R}} v^3 dx, \quad u, v \in E,$$

and

$$\Phi(\mathbf{u}) = I_1(u) + I_2(v) - \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v dx, \quad \mathbf{u} \in \mathbb{E}.$$

We also write

$$G_\beta(\mathbf{u}) = \frac{1}{4} \int_{\mathbb{R}} u^4 dx + \frac{1}{6} \int_{\mathbb{R}} v^3 dx + \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v dx, \quad \mathbf{u} \in \mathbb{E},$$

and using this notation we can rewrite the energy functional

$$\Phi(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 - G_\beta(\mathbf{u}), \quad \mathbf{u} \in \mathbb{E}.$$

Definition 1. We say that $\mathbf{u} \in \mathbb{E}$ is a non-trivial bound state of (3) if \mathbf{u} is a non-trivial critical point of Φ . A bound state $\tilde{\mathbf{u}}$ is called ground state if its energy is minimal among all the non-trivial bound states, namely

$$\Phi(\tilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E} \setminus \{\mathbf{0}\}, \Phi'(\mathbf{u}) = 0\}. \quad (4)$$

3. NEHARI MANIFOLD AND KEY RESULTS

We will work mainly in \mathbb{H} . Setting

$$\Psi(\mathbf{u}) = (\nabla \Phi(\mathbf{u}) | \mathbf{u}) = (I'_1(u) | u) + (I'_2(v) | v) - \frac{3}{2}\beta \int_{\mathbb{R}} u^2 v dx,$$

we define the corresponding Nehari manifold

$$\mathcal{N} = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

Then, one has that

$$(\nabla \Psi(\mathbf{u}) | \mathbf{u}) = -\|\mathbf{u}\|^2 - \int_{\mathbb{R}} u^4 dx < 0, \quad \forall \mathbf{u} \in \mathcal{N}, \quad (5)$$

thus \mathcal{N} is a smooth manifold locally near any point $\mathbf{u} \neq \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$. Moreover, $\Phi''(\mathbf{0}) = I''_1(0) + I''_2(0)$ is positive definite, so we infer that $\mathbf{0}$ is a strict minimum for Φ . As a consequence, $\mathbf{0}$ is an isolated point of the set $\{\Psi(\mathbf{u}) = 0\}$, proving that, on the one hand \mathcal{N} is a smooth complete manifold of codimension 1, and on the other hand there exists a constant $\rho > 0$ so that

$$\|\mathbf{u}\|^2 > \rho, \quad \forall \mathbf{u} \in \mathcal{N}. \quad (6)$$

Furthermore, (5) and (6) plainly imply that $\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\}$ is a critical point of Φ if and only if $\mathbf{u} \in \mathcal{N}$ is a critical point of Φ constrained on \mathcal{N} .

Remarks 2. (i) *By the previous arguments, the Nehari manifold \mathcal{N} is a natural constraint of Φ . Also, it is relevant to point out that working on the Nehari manifold, the functional Φ takes the form:*

$$\Phi|_{\mathcal{N}}(\mathbf{u}) = \frac{1}{6}\|\mathbf{u}\|^2 + \frac{1}{12} \int_{\mathbb{R}^n} u^4 dx =: F(\mathbf{u}), \quad (7)$$

and by using (6) into (7) we have

$$\Phi(\mathbf{u}) \geq \frac{1}{6}\|\mathbf{u}\|^2 > \frac{1}{6}\rho \quad \forall \mathbf{u} \in \mathcal{N}. \quad (8)$$

Therefore, (8) shows that the functional Φ is bounded from below on \mathcal{N} , so one can try to minimize it on the Nehari manifold.

- (ii) *With respect to the Palais-Smale (PS for short) condition, we recall that in the one dimensional case, one cannot expect a compact embedding of E into $L^q(\mathbb{R})$ for $2 < q < \infty$. Indeed, working on H (the radial or even case) is not true too; see [28, Remarque I.1]. However, we will show that for a PS sequence we can find a subsequence for which the weak limit is a solution. This fact jointly with some properties of the Schwarz symmetrization will permit us to prove the existence of positive even ground states in Theorem 6.*

Due to the lack of compactness mentioned above in Remarks 2-(ii), we state a measure theory result given in [29] that we will use in the proof of Theorem 6.

Lemma 3. *If $2 < q < \infty$, there exists a constant $C > 0$ so that*

$$\int_{\mathbb{R}^n} |u|^q dx \leq C \left(\sup_{z \in \mathbb{R}} \int_{|x-z|<1} |u(x)|^2 dx \right)^{\frac{q-2}{2}} \|u\|_E^2, \quad \forall u \in E. \quad (9)$$

See [12] for an extension of this lemma in fractional Sobolev spaces, and an application of it in a fractional system of NLS equations.

Let V denotes the unique positive even solution of $-v'' + v = v^2$, $v \in H$; see [25]. Setting

$$V_2(x) = 2\lambda_2 V(\sqrt{\lambda_2}x) = \frac{3\lambda_2}{\cosh^2\left(\frac{\sqrt{\lambda_2}}{2}x\right)}, \quad (10)$$

one has that V_2 is the unique positive solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2$ in H . Hence $\mathbf{v}_2 := (0, V_2)$ is a particular solution of (3) for any $\beta \in \mathbb{R}$, and moreover, it is the unique non-negative semi-trivial solution of (3). We also define the corresponding Nehari manifold,

$$\mathcal{N}_2 = \left\{ v \in H : (I'_2(v)|v) = 0 \right\} = \left\{ v \in H : \|v\|_2^2 - \frac{1}{2} \int_{\mathbb{R}} v^3 dx = 0 \right\}.$$

Let us denote $T_{\mathbf{v}_2}\mathcal{N}$ the tangent space to \mathcal{N} on \mathbf{v}_2 . Since

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \iff (V_2|h_2) = \frac{3}{4} \int_{\mathbb{R}} V_2^2 h_2 dx,$$

it follows that

$$(h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \iff h_2 \in T_{V_2}\mathcal{N}_2. \quad (11)$$

Proposition 4. *There exists $\Lambda > 0$ such that:*

- (i) *if $\beta < \Lambda$, then \mathbf{v}_2 is a strict local minimum of Φ constrained on \mathcal{N} ,*
- (ii) *for any $\beta > \Lambda$, then \mathbf{v}_2 is a saddle point of Φ constrained on \mathcal{N} . Moreover, $\inf_{\mathcal{N}} \Phi < \Phi(\mathbf{v}_2)$.*

Proof. (i) We define

$$\Lambda = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}} V_2 \varphi^2 dx}. \quad (12)$$

One has that for $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}$,

$$\Phi''(\mathbf{v}_2)[\mathbf{h}]^2 = \|h_1\|_1^2 + I_2''(V_2)[h_2]^2 - \beta \int_{\mathbb{R}} V_2 h_1^2 dx. \quad (13)$$

Let us take $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N}$, by (11) $h_2 \in T_{V_2}\mathcal{N}_2$, then using that V_2 is the minimum of I_2 on \mathcal{N}_2 , there exists a constant $c > 0$ so that

$$I_2''(V_2)[h_2]^2 \geq c \|h_2\|_2^2. \quad (14)$$

Let h_1 be a function with $\|h_1\|_1^2 = \Lambda \int_{\mathbb{R}} V_2 h_1^2 dx$, i.e., which exists since the infimum Λ defined by (12) is achieved, using this fact jointly with (13) and $\beta < \Lambda$, there exists another constant $c_1 > 0$ so that,

$$\Phi''(\mathbf{v}_2)[\mathbf{h}]^2 \geq c_1 \|h_1\|_1^2 + c_2 \|h_2\|_2^2. \quad (15)$$

Notice that $\Phi'(\mathbf{v}_2) = 0$ implies that $D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2 = \Phi''(\mathbf{v}_2)[\mathbf{h}]^2$ for all $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}$, and thus using (15) we infer that \mathbf{v}_2 is a local strict minimum of Φ on \mathcal{N} .

(ii) According to (11), $\mathbf{h} = (h_1, 0) \in T_{\mathbf{v}_2}\mathcal{N}$ for any $h_1 \in H$. We have that, for $\beta > \Lambda$, there exists $\tilde{h} \in H$ with

$$\Lambda < \frac{\|\tilde{h}\|_1^2}{\int_{\mathbb{R}} V_2 \tilde{h}^2 dx} < \beta,$$

thus, taking $\mathbf{h}_0 = (\tilde{h}, 0) \in T_{\mathbf{v}_2}\mathcal{N}$, by (13) we find

$$\Phi''(\mathbf{v}_2)[\mathbf{h}_0]^2 = \|\tilde{h}\|_1^2 - \beta \int_{\mathbb{R}} V_2 \tilde{h}^2 dx < 0,$$

finishing the proof taking $\Lambda = \Lambda$. ■

Remark 5. If one consider $\lambda_1 = \lambda_2$ as in [17], taking $\mathbf{h}_0 = (V_2, 0) \in T_{\mathbf{v}_2}\mathcal{N}$ in the proof of Proposition 4-(ii), one finds that

$$\Phi''(\mathbf{v}_2)[\mathbf{h}_0]^2 = \|V_2\|_2^2 - \beta \int_{\mathbb{R}} V_2^3 dx = (1 - 2\beta)\|V_2\|_2^2 < 0 \quad \text{provided} \quad \beta > \frac{1}{2}.$$

See also Remark 7.

4. MAIN RESULTS

4.1. Existence of Ground states. Concerning the existence of ground state solutions of (3), the first result is the following.

Theorem 6. Suppose that $\beta > \Lambda$, then System (3) has a positive even ground state $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$.

Proof. We divide the proof into two steps. In the first step, we prove that $\inf_{\mathcal{N}} \Phi$ is achieved at some positive function $\tilde{\mathbf{u}} \in \mathbb{H}$, while in the second step, we show that $\tilde{\mathbf{u}}$ is indeed a ground state, i.e.,

$$\Phi(\tilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E}, \Phi'(\mathbf{u}) = 0\}.$$

Step 1. By the Ekeland's variational principle; see [19], there exists a PS sequence $\{\mathbf{u}_k\}_{k \in \mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \rightarrow c = \inf_{\mathcal{N}} \Phi \tag{16}$$

$$\nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \rightarrow 0. \tag{17}$$

By (7), easily one finds that $\{\mathbf{u}_k\}$ is a bounded sequence on \mathbb{E} , and relabeling, we can assume that $\mathbf{u}_k \rightharpoonup \mathbf{u}$ weakly in \mathbb{E} , $\mathbf{u}_k \rightarrow \mathbf{u}$ strongly in $\mathbb{L}_{loc}^q(\mathbb{R}) = L_{loc}^q(\mathbb{R}) \times L_{loc}^q(\mathbb{R})$ for every $1 \leq q < \infty$ and $\mathbf{u}_k \rightarrow \mathbf{u}$ a.e. in \mathbb{R}^2 . Moreover, the constrained gradient $\nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) = \Phi'(\mathbf{u}_k) - \eta_k \Psi'(\mathbf{u}_k) \rightarrow 0$, where η_k is the corresponding Lagrange multiplier. Taking the scalar product with \mathbf{u}_k and recalling that $(\Phi'(\mathbf{u}_k) | \mathbf{u}_k) = \Psi(\mathbf{u}_k) = 0$, we find that $\eta_k(\Psi'(\mathbf{u}_k) | \mathbf{u}_k) \rightarrow 0$ and this jointly with (5)-(6) imply that $\eta_k \rightarrow 0$. Since in addition $\|\Psi'(\mathbf{u}_k)\| \leq C < +\infty$, we deduce that $\Phi'(\mathbf{u}_k) \rightarrow 0$.

Let us define $\mu_k = u_k^2 + v_k^2$, where $\mathbf{u}_k = (u_k, v_k)$. We *claim* that there is no evanescence, i.e., exist $R, C > 0$ so that

$$\sup_{z \in \mathbb{R}} \int_{|z| < R} \mu_k \geq C > 0, \quad \forall k \in \mathbb{N}. \tag{18}$$

On the contrary, if we suppose

$$\sup_{z \in \mathbb{R}} \int_{|z| < R} \mu_k \rightarrow 0,$$

by Lemma 3, applied in a similar way as in [12], we find that $\mathbf{u}_k \rightarrow \mathbf{0}$ strongly in $\mathbb{L}^q(\mathbb{R})$ for any $2 < q < \infty$, and as a consequence the weak limit $\mathbf{u}^* \equiv \mathbf{0}$. This is a contradiction since $\mathbf{u}_k \in \mathcal{N}$, and by (7), (8), (16) there holds

$$0 < \frac{1}{7}\rho < c + o_k(1) = \Phi(\mathbf{u}_k) = F(\mathbf{u}_k), \quad \text{with } o_k(1) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

hence (18) is true and the *claim* is proved.

We observe that we can find a sequence of points $\{z_k\} \subset \mathbb{R}^2$ so that by (18), the translated sequence $\bar{\mu}_k(x) = \mu_k(x + z_k)$ satisfies

$$\liminf_{k \rightarrow \infty} \int_{B_R(0)} \bar{\mu}_k \geq C > 0.$$

Taking into account that $\bar{\mu}_k \rightarrow \bar{\mu}$ strongly in $L^1_{loc}(\mathbb{R})$, we obtain that $\bar{\mu} \not\equiv 0$. Therefore, defining $\bar{\mathbf{u}}_k(x) = \mathbf{u}_k(x + z_k)$, we have that $\bar{\mathbf{u}}_k$ is also a PS sequence of Φ on \mathcal{N} , in particular the weak limit of $\bar{\mathbf{u}}_k$, denoted by $\bar{\mathbf{u}}$, is a non-trivial critical point of Φ constrained on \mathcal{N} , so $\bar{\mathbf{u}} \in \mathcal{N}$. Thus, using (7) again, we find

$$\begin{aligned} \Phi(\bar{\mathbf{u}}) &= F(\bar{\mathbf{u}}) \\ &\leq \liminf_{k \rightarrow \infty} F(\bar{\mathbf{u}}_k) \\ &= \liminf_{k \rightarrow \infty} \Phi(\bar{\mathbf{u}}_k) = c. \end{aligned}$$

Furthermore, by Proposition 4-(ii) we know that necessarily $\Phi(\bar{\mathbf{u}}) < \Phi(\mathbf{v}_2)$.

Taking into account that $\bar{\mathbf{u}} \in \mathcal{N}$, and the maximum principle, then $\bar{v} > 0$, thus it is not difficult to show $\tilde{\mathbf{u}} = |\bar{\mathbf{u}}| = (|\bar{u}|, |\bar{v}|) = (\bar{u}, \bar{v}) \in \mathcal{N}$ with

$$\Phi(\tilde{\mathbf{u}}) = \Phi(\bar{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\}, \quad (19)$$

so we have $\tilde{\mathbf{u}} \geq \mathbf{0}$. Finally, by the maximum principle applied to the first equation and the fact that $\Phi(\tilde{\mathbf{u}}) < \Phi(\mathbf{v}_2)$, we get $\tilde{\mathbf{u}} > \mathbf{0}$.

Step 2. Assume, for a contradiction, there exists $\mathbf{w}_0 \in \mathbb{E}$ a non-trivial critical point of Φ such that

$$\Phi(\mathbf{w}_0) < \Phi(\tilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\}. \quad (20)$$

Setting $\mathbf{w} = |\mathbf{w}_0|$ there holds

$$\Phi(\mathbf{w}) = \Phi(\mathbf{w}_0), \quad \Psi(\mathbf{w}) = \Psi(\mathbf{w}_0). \quad (21)$$

For $\mathbf{w} = (w_1, w_2)$, we set $\mathbf{w}^* = (w_1^*, w_2^*)$, where w_j^* is the Schwartz symmetric function associated to $w_j \geq 0$; $j = 1, 2$. Then by the classical properties of the Schwartz symmetrization; see for instance [24], there hold

$$\|\mathbf{w}^*\|^2 \leq \|\mathbf{w}\|^2, \quad G_\beta(\mathbf{w}^*) \geq G_\beta(\mathbf{w}), \quad (22)$$

thus, in particular, $\Psi(\mathbf{w}^*) \leq \Psi(\mathbf{w})$. Using the second identity of (21) and the fact that \mathbf{w}_0 is a critical point of Φ , we get $\Psi(\mathbf{w}) = \Psi(\mathbf{w}_0) = 0$. Furthermore, there exists a unique $t_0 > 0$ so that $t_0 \mathbf{w}^* \in \mathcal{N}$. In fact, t_0 comes from $\Psi(t_0 \mathbf{w}^*) = 0$, i.e.,

$$\|\mathbf{w}^*\|^2 = t_0^2 \int_{\mathbb{R}} (w_1^*)^4 dx + t_0 \left(\frac{1}{2} \int_{\mathbb{R}} (w_2^*)^3 dx + \frac{3}{2} \beta \int_{\mathbb{R}} (w_1^*)^2 w_2^* dx \right), \quad (23)$$

then using that $\Psi(\mathbf{w}) = 0$, (22)-(23) and the fact that $\mathbf{w} > \mathbf{0}$ and $t_0 > 0$ we find

$$\begin{aligned} & \int_{\mathbb{R}} w_1^4 dx + \frac{1}{2} \int_{\mathbb{R}} w_2^3 dx + \frac{3}{2} \beta \int_{\mathbb{R}} w_1^2 w_2 dx \\ & \geq t_0^2 \int_{\mathbb{R}} w_1^4 dx + t_0 \left(\frac{1}{2} \int_{\mathbb{R}} w_2^3 dx + \frac{3}{2} \beta \int_{\mathbb{R}} w_1^2 w_2 dx \right). \end{aligned}$$

Thus, clearly $t_0 \leq 1$, and as a consequence,

$$\Phi(t_0 \mathbf{w}^*) = \frac{1}{6}t_0^2 \|\mathbf{w}^*\|^2 + \frac{1}{12}t_0^4 \int_{\mathbb{R}} (w_1^*)^4 dx \leq \frac{1}{6} \|\mathbf{w}\|^2 + \frac{1}{12} \int_{\mathbb{R}} w_1^4 dx = \Phi(\mathbf{w}). \quad (24)$$

Therefore, inequalities (24), (20) and the first identity of (21) yield

$$\Phi(t_0 \mathbf{w}^*) \leq \Phi(\mathbf{w}) < \Phi(\tilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\},$$

which is a contradiction because $t_0 \mathbf{w}^* \in \mathcal{N}$. ■

Remark 7. As we anticipated at the introduction, see also Remark 5, in the setting by [17], $\lambda_1 = \lambda_2$ and $\beta > \frac{1}{2}$, we have found positive even ground state solutions in contrast with the non-negative bound states founded by [17].

The last result in this subsection deals with the existence of positive ground states of (3) not only for $\beta > \Lambda$, but also for $0 < \beta \leq \Lambda$, at least for λ_2 large enough.

Theorem 8. *There exists $\Lambda_2 > 0$ such that if $\lambda_2 > \Lambda_2$, System (3) has an even ground state $\tilde{\mathbf{u}} > \mathbf{0}$ for every $\beta > 0$.*

Proof. Arguing in the same way as in the proof of Theorem 6, we initially have that there exists an even ground state $\tilde{\mathbf{u}} \geq \mathbf{0}$. Moreover, in Theorem 6 for $\beta > \Lambda$ we proved that $\tilde{\mathbf{u}} > \mathbf{0}$. Now we need to show that for $\beta \leq \Lambda$ indeed $\tilde{\mathbf{u}} > \mathbf{0}$ which follows by the maximum principle provided $\tilde{\mathbf{u}} \neq \mathbf{v}_2$. Taking into account Proposition (4)-(i), \mathbf{v}_2 is a strict local minimum, but this does not allow us to prove that $\tilde{\mathbf{u}} \neq \mathbf{v}_2$. The new idea here consists on proving the existence of a function $\mathbf{u}_1 = (u_1, v_1) \in \mathcal{N}$ with $\Phi(\mathbf{u}_1) < \Phi(\mathbf{v}_2)$. To do so, since $\mathbf{v}_2 = (0, V_2)$ is a local minimum of Φ on \mathcal{N} provided $0 < \beta < \Lambda$, we cannot find \mathbf{u}_1 in a neighborhood of \mathbf{v}_2 on \mathcal{N} . Thus, we define $\mathbf{u}_1 = t(V_2, V_2)$ where $t > 0$ is the unique value so that $\mathbf{u}_1 \in \mathcal{N}$.

Notice that $t > 0$ is given by $\Psi(\mathbf{u}_1) = 0$, i.e.,

$$\|(V_2, V_2)\|^2 = t^2 \int_{\mathbb{R}} V_2^4 dx + \frac{1}{2}t(1 + 3\beta) \int_{\mathbb{R}} V_2^3 dx. \quad (25)$$

Moreover,

$$\|(V_2, V_2)\|^2 = 2\|V_2\|_2^2 + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2 dx = \int_{\mathbb{R}} V_2^3 dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2 dx, \quad (26)$$

hence, substituting (26) into (25) we get

$$t^2 \int_{\mathbb{R}} V_2^4 dx + \frac{1}{2}t(1 + 3\beta) \int_{\mathbb{R}} V_2^3 dx = \int_{\mathbb{R}} V_2^3 dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2 dx,$$

therefore, dividing the above expression by the L^1 norm of V_2^3 , using that

$$\int_{\mathbb{R}} \cosh^{-8}(x) dx = \frac{32}{35}, \quad \int_{\mathbb{R}} \cosh^{-6}(x) dx = \frac{16}{15}, \quad \int_{\mathbb{R}} \cosh^{-4}(x) dx = \frac{4}{3},$$

and the definition of V_2 by (10), we find

$$\frac{18}{7}\lambda_2 t^2 + \frac{1}{2}t(1 + 3\beta) - \left(1 + 5\frac{\lambda_1 - \lambda_2}{12\lambda_2}\right) = 0. \quad (27)$$

The energies of $\mathbf{u}_1, \mathbf{v}_2$ are given by

$$\Phi(t(V_2, V_2)) = \frac{1}{6}t^2 \left(\int_{\mathbb{R}} V_2^3 dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2 dx \right) + \frac{1}{12}t^4 \int_{\mathbb{R}} V_2^4 dx,$$

$$\Phi(\mathbf{v}_2) = \frac{1}{12} \int_{\mathbb{R}} V_2^3 dx.$$

Thus, we want to prove that for the unique $t > 0$ given by (26) we have

$$\frac{1}{6}t^2 \left(\int_{\mathbb{R}} V_2^3 dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2 dx \right) + \frac{1}{12}t^4 \int_{\mathbb{R}} V_2^4 dx < \frac{1}{12} \int_{\mathbb{R}} V_2^3 dx,$$

then arguing as for (27), it is sufficient to prove that the following inequality holds

$$\frac{18}{7}\lambda_2 t^4 + t^2 \left(2 + 5 \frac{\lambda_1 - \lambda_2}{6\lambda_2} \right) - 1 < 0. \quad (28)$$

Using (27) and the fact that $2 + 5 \frac{\lambda_1 - \lambda_2}{6\lambda_2} > 0$ for every $\lambda_1, \lambda_2 > 0$, fixed $\beta > 0$ we have that (28) is satisfied provided λ_2 is sufficiently large, namely $\lambda_2 > \Lambda_2 > 0$, proving that $\Phi(\mathbf{u}_1) < \Phi(\mathbf{v}_2)$ which concludes the result. ■

4.2. Existence of Bound states. In this subsection we establish existence of bound states to (3). The first theorem deals with a perturbation framework, in which we suppose that $\beta = \varepsilon \tilde{\beta}$, with $\tilde{\beta}$ fixed and independent of ε . Note that $\tilde{\beta}$ can be negative, and $0 < \varepsilon \ll 1$. Then we rewrite the energy functional Φ as Φ_ε to emphasize its dependence on ε ,

$$\Phi_\varepsilon(\mathbf{u}) = \Phi_0(\mathbf{u}) - \frac{1}{2}\varepsilon \tilde{\beta} \int_{\mathbb{R}} u^2 v dx,$$

where $\Phi_0 = I_1 + I_2$.

Let us set $\mathbf{u}_0 = (U_1, V_2)$, where V_2 is given by (10) and U_1 is the unique positive solution of $-u'' + \lambda_1 u = u^3$ in H ; see [15, 25]. This function U_1 has the following explicit expression,

$$U_1(x) = \frac{\sqrt{2\lambda_1}}{\cosh(\sqrt{\lambda_1}x)}. \quad (29)$$

Note also that U_1 satisfies the following identity,

$$\|U_1\|_1 = \inf_{u \in H \setminus \{0\}} \frac{\|u\|_1^2}{\left(\int_{\mathbb{R}} u^4 dx \right)^{1/2}}. \quad (30)$$

Theorem 9. *There exists $\varepsilon_0 > 0$ so that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon \tilde{\beta}$, System (3) has an even bound state \mathbf{u}_ε with $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$ as $\varepsilon \rightarrow 0$. Moreover, if $\beta > 0$ then $\mathbf{u}_\varepsilon > \mathbf{0}$.*

In order to prove this result, we follow some ideas of [13, Theorem 4.2] with appropriate modifications.

Proof of Theorem 9. It is well known that U_1 and V_2 are non-degenerate critical points of I_1 and I_2 on H respectively; [25]. Plainly, \mathbf{u}_0 is a non-degenerate critical point of Φ_0 acting on \mathbb{H} . Then, by the Local Inversion Theorem, there exists a critical point \mathbf{u}_ε of Φ_ε for any $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small; see [8] for more details. Moreover, $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$ on \mathbb{H} as $\varepsilon \rightarrow 0$. To complete the proof it remains to show that if $\beta > 0$, then $\mathbf{u}_\varepsilon > \mathbf{0}$.

Let us denote the positive part $\mathbf{u}_\varepsilon^+ = (u_\varepsilon^+, v_\varepsilon^+)$ and the negative part $\mathbf{u}_\varepsilon^- = (u_\varepsilon^-, v_\varepsilon^-)$. By (30) we have

$$\|u_\varepsilon^\pm\|_1^2 \geq \|U_1\|_1 \left(\int_{\mathbb{R}} (u_\varepsilon^\pm)^4 dx \right)^{1/2}. \quad (31)$$

Multiplying the second equation of (3) by v_ε^- and integrating on \mathbb{R} one obtains

$$\|v_\varepsilon^-\|_2^2 = \int_{\mathbb{R}} (v_\varepsilon^-)^3 dx + \varepsilon \tilde{\beta} \int_{\mathbb{R}} (u_\varepsilon)^2 v_\varepsilon^- dx \leq 0, \quad (32)$$

thus $\|v_\varepsilon^-\|_2 = 0$ which implies $v_\varepsilon = v_\varepsilon^+ \geq 0$. Furthermore, $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$ implies $v_\varepsilon \rightarrow V_2$, which jointly with the maximum principle gives $v_\varepsilon > 0$ provided ε is sufficiently small.

Multiplying now the first equation of (3) by u_ε^\pm and integrating on \mathbb{R} one obtains

$$\begin{aligned} \|u_\varepsilon^\pm\|_1^2 &= \int_{\mathbb{R}} (u_\varepsilon^\pm)^4 dx + \varepsilon \tilde{\beta} \int_{\mathbb{R}} (u_\varepsilon^\pm)^2 v_\varepsilon dx \\ &\leq \int_{\mathbb{R}} (u_\varepsilon^\pm)^4 dx + \varepsilon \tilde{\beta} \left(\int_{\mathbb{R}} (u_\varepsilon^\pm)^4 dx \right)^{1/2} \left(\int_{\mathbb{R}} v_\varepsilon^2 dx \right)^{1/2}. \end{aligned}$$

This, jointly with (31), yields

$$\|u_\varepsilon^\pm\|_1^2 \leq \frac{\|u_\varepsilon^\pm\|_1^4}{\|U_1\|_1^2} + \varepsilon \theta_\varepsilon \frac{\|u_\varepsilon^\pm\|_1^2}{\|U_1\|_1}, \quad (33)$$

where

$$\theta_\varepsilon = \tilde{\beta} \left(\int_{\mathbb{R}^n} v_\varepsilon^2 \right)^{1/2}.$$

Hence, if $\|u_{1\varepsilon}^\pm\| > 0$, one infers

$$\|u_\varepsilon^\pm\|_1^2 \geq \|U_1\|_1^2 + o(1), \quad (34)$$

where $o(1) = o_\varepsilon(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Using again $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$, then $u_\varepsilon \rightarrow U_1 > 0$, as a consequence, for ε small enough, $\|u_\varepsilon^+\| > 0$. Thus (34) gives

$$\|\mathbf{u}_\varepsilon^+\|^2 = \|u_\varepsilon^+\|_1^2 + \|v_\varepsilon^+\|_2^2 \geq \|U_1\|_1^2 + o(1). \quad (35)$$

Now, suppose for a contradiction, that $\|\mathbf{u}_\varepsilon^-\|_1 > 0$. Then as for (35), one obtains

$$\|\mathbf{u}_\varepsilon^-\|^2 = \|u_\varepsilon^-\|_1^2 + \|v_\varepsilon^-\|_2^2 \geq \|U_1\|_1^2 + o(1). \quad (36)$$

On one hand, using (35)-(36), we find

$$\begin{aligned} \Phi(\mathbf{u}_\varepsilon) &= \frac{1}{6} \|\mathbf{u}_\varepsilon\|^2 + \frac{1}{12} \int_{\mathbb{R}} u_\varepsilon^4 dx \\ &= \frac{1}{6} [\|\mathbf{u}_\varepsilon^+\|^2 + \|\mathbf{u}_\varepsilon^-\|^2] + \frac{1}{12} \int_{\mathbb{R}} [(u_\varepsilon^+)^4 + (u_\varepsilon^-)^4] dx \\ &\geq \frac{1}{6} \|\mathbf{u}_0\|^2 + \frac{1}{6} \|U_1\|_1^2 + \frac{1}{12} \int_{\mathbb{R}} U_1^4 dx + o(1). \end{aligned} \quad (37)$$

On the other hand, since $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$ we have

$$\Phi(\mathbf{u}_\varepsilon) = \frac{1}{6} \|\mathbf{u}_\varepsilon\|^2 + \frac{1}{12} \int_{\mathbb{R}} u_\varepsilon^4 dx \rightarrow \frac{1}{6} \|\mathbf{u}_0\|^2 + \frac{1}{12} \int_{\mathbb{R}} U_1^4 dx, \quad (38)$$

which is in contradiction with (37), proving that $u_\varepsilon \geq 0$.

In conclusion, we have proved that $v_\varepsilon > 0$ and $u_\varepsilon \geq 0$. To prove the positivity of u_ε , using once more that $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0$, and $\beta = \varepsilon \tilde{\beta} \geq 0$ we can apply the maximum principle to the first equation of (3), which implies that $u_\varepsilon > 0$, and finally, $\mathbf{u}_\varepsilon > \mathbf{0}$. ■

From the existence of a positive ground state established in Theorem 6 for $\beta > \Lambda$, and more precisely in Theorem 8 for $\beta > 0$, provided λ_2 is sufficiently large, we can show the existence of a different positive bound state of (3) in the following.

Theorem 10. *In the hypotheses of Theorem 8 and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.*

Proof. The positive ground state $\tilde{\mathbf{u}}$ founded in Theorem 8 satisfies $\Phi(\tilde{\mathbf{u}}) < \Phi(\mathbf{v}_2)$ and even more, if $\beta < \Lambda$ by Proposition 4, \mathbf{v}_2 is a strict local minimum of Φ constrained on \mathcal{N} . As a consequence, we have the Mountain Pass (MP in short) geometry between $\tilde{\mathbf{u}}$ and \mathbf{v}_2 on \mathcal{N} . We define the set of all continuous paths joining $\tilde{\mathbf{u}}$ and \mathbf{v}_2 on the Nehari manifold by

$$\Gamma = \{\gamma : [0, 1] \rightarrow \mathcal{N} \text{ continuous} \mid \gamma(0) = \tilde{\mathbf{u}}, \gamma(1) = \mathbf{v}_2\}.$$

Thanks to the MP Theorem by Ambrosetti-Rabinowitz; [9], there exists a PS sequence $\{\mathbf{u}_k\} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \rightarrow c = \inf_{\mathcal{N}} \Phi, \quad \nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \rightarrow 0,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\gamma(t)). \quad (39)$$

Plainly, by (7) the sequence $\{\mathbf{u}_k\}$ is bounded on \mathbb{H} , and we obtain a weakly convergent subsequence $\mathbf{u}_k \rightharpoonup \mathbf{u}^* \in \mathcal{N}$.

The difficulty of the lack of compactness, due to work in the one dimensional case (see Remark 2-(ii)), can be circumvented in a similar way as in the proof of Theorem 6, so we omit the full detail for short. Thus, we find that the weak limit $\mathbf{u}^* = (u^*, v^*)$ is an even bound state of (3), and clearly, $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

It remains to prove that $\mathbf{u}^* > \mathbf{0}$. To do so, let us introduce the following problem

$$\begin{cases} -u'' + \lambda_1 u &= (u^+)^3 + \beta u^+ v \\ -v'' + \lambda_2 v &= \frac{1}{2} v^2 + \frac{1}{2} \beta (u^+)^2. \end{cases} \quad (40)$$

By the maximum principle every nontrivial solution $\mathbf{u} = (u, v)$ of (40) has the second component $v > 0$ and the first one $u \geq 0$. Let us define its energy functional

$$\Phi^+(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 - G_\beta(u^+, v),$$

and consider the corresponding Nehari manifold

$$\mathcal{N}^+ = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : (\nabla \Phi^+(\mathbf{u}) | \mathbf{u}) = 0\}.$$

Also, we denote

$$I_1^+(u) = \frac{1}{2} \|u\|_1^2 - \frac{1}{4} \int_{\mathbb{R}} (u^+)^4 dx.$$

It is not very difficult to show that the properties proved for Φ and \mathcal{N} still hold for Φ^+ and \mathcal{N}^+ . Unfortunately, Φ^+ is not \mathcal{C}^2 , thus Proposition 4-(i) does not holds directly for Φ^+ . To solve this difficulty, we are going to prove that \mathbf{v}_2 is a strict local minimum of Φ^+ constrained on \mathcal{N}^+ without using the second derivative of the functional. Note that in a similar way as in (11), there holds

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2} \mathcal{N}^+ \iff h_2 \in T_{\mathbf{v}_2} \mathcal{N}_2. \quad (41)$$

Taking $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}^+$ with $\|\mathbf{h}\| = 1$, we consider $\mathbf{v}_\varepsilon = (\varepsilon h_1, V_2 + \varepsilon h_2)$. Plainly, there exists a unique $t_\varepsilon > 0$ so that $t_\varepsilon \mathbf{v}_\varepsilon \in \mathcal{N}^+$. Thus, we want to prove there exists $\varepsilon_1 > 0$ so that

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \Phi^+(\mathbf{v}_2) \quad \forall 0 < \varepsilon < \varepsilon_1.$$

It is convenient to distinguish if $h_1 = 0$ or not. In the former case, $h_1 = 0$, $\mathbf{v}_\varepsilon = (0, V_2 + \varepsilon h_2)$. Hence $t_\varepsilon \mathbf{v}_\varepsilon \in \mathcal{N}^+ \Leftrightarrow t_\varepsilon(V_2 + \varepsilon h_2) \in \mathcal{N}_2$. Furthermore,

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) = I_2(t_\varepsilon(V_2 + \varepsilon h_2)) > I_2(V_2) = \Phi(\mathbf{v}_2) = \Phi^+(\mathbf{v}_2), \quad (42)$$

where the previous inequality holds because V_2 is a strict local minimum of I_2 on \mathcal{N}_2 .

Let us now consider the case $h_1 \neq 0$. There holds

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) = I_2(t_\varepsilon(V_2 + \varepsilon h_2)) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2} \beta \varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2) dx. \quad (43)$$

By (42) and (43) it follows,

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \Phi^+(\mathbf{v}_2) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2} \beta \varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2) dx. \quad (44)$$

To finish, it is sufficient to show that

$$\mathcal{J}(t_\varepsilon \mathbf{v}_\varepsilon) := I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2} \beta \varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2) dx > 0 \quad \forall 0 < \varepsilon < \varepsilon_1.$$

Let $\alpha < 1$ be such that $\alpha > \frac{\beta}{\Lambda}$. By (12) and $\beta < \Lambda$ there holds

$$\beta \int_{\mathbb{R}} V_2 (h_1^+)^2 dx < \alpha \|h_1\|_1^2,$$

then for ε_1 smaller than before (if necessary) we have

$$\beta \int_{\mathbb{R}} (V_2 + \varepsilon h_2) (h_1^+)^2 dx < \alpha \|h_1\|_1^2 \quad \forall 0 < \varepsilon < \varepsilon_1. \quad (45)$$

Using (45) and the Sobolev inequality, we obtain

$$\mathcal{J}(t_\varepsilon \mathbf{v}_\varepsilon) > \frac{1}{2} t_\varepsilon^2 \varepsilon^2 \|h_1\|_1^2 (1 - t_\varepsilon \alpha - c t_\varepsilon^2 \varepsilon^2), \quad \text{for a constant } c > 0.$$

Now, taking into account that $t_\varepsilon \rightarrow 1$ as $\varepsilon \searrow 0$, we infer there exists a constant $c_0 > 0$ so that

$$\mathcal{J}(t_\varepsilon \mathbf{v}_\varepsilon) > \varepsilon^2 c_0 \|h_1\|_1^2. \quad (46)$$

Finally, by (44), (46) it follows that

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \varepsilon^2 c_0 \|h_1\|_1^2 + \Phi^+(\mathbf{v}_2) > \Phi^+(\mathbf{v}_2),$$

which proves that \mathbf{v}_2 is a strict local minimum for Φ^+ on \mathcal{N}^+ .

From the preceding arguments, it follows that Φ^+ has a MP critical point $\mathbf{u}^* \in \mathcal{N}^+$, which gives rise to a solution of (40). In particular, one finds that $u, v \geq 0$. In addition, since \mathbf{u}^* is a MP critical point, one has that $\Phi(\mathbf{u}^*) = \Phi^+(\mathbf{u}^*) > \Phi^+(\mathbf{v}_2) = \Phi(\mathbf{v}_2) > 0$, which implies $u^* \geq 0$ with $u^* \not\equiv 0$, and by the maximum principle applied to each single equation we get $u^*, v^* > 0$, hence $\mathbf{u}^* > \mathbf{0}$. ■

In view of Theorems 8, 10, some remarks are in order.

- Remarks 11.** (i) Following the proof of Theorem 10, a natural question is what happens in the limit case $\beta = \Lambda$. In that case \mathbf{u}^* could coincide with \mathbf{v}_2 which is non-negative, but not positive. Indeed, this is our conjecture in view of the second equation by (52); see also Figure 16.
- (ii) In the hypotheses of Theorems 8, 10 we have found the coexistence of two positive solutions, the ground state $\tilde{\mathbf{u}}$ in Theorem 8 and the bound state \mathbf{u}^* in Theorem 10, proving a non-uniqueness result of positive solutions to (3). This is a great difference with the more studied system of coupled NLS equations

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1 \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2, \end{cases}$$

(see for instance [4, 5, 6, 10, 11, 13, 20, 22, 23, 26, 27, 30, 32, 33, 34, 35] and the references therein) for which it is known that there is uniqueness of positive solutions, under appropriate conditions on the parameters including the case $\beta > 0$ small; see more specifically [22, 35]. Indeed, for $\beta > 0$ small, the ground state is not positive, and it is given by one of the two semi-trivial solutions $(U^{(1)}, 0)$ or $(0, U^{(2)})$ depending on if $\Phi(U^{(1)}, 0)$ is lower or greater than $\Phi(0, U^{(2)})$ which plainly corresponds to $\lambda_1^{2-\frac{n}{2}} \mu_2 < \lambda_2^{2-\frac{n}{2}} \mu_1$ or $\lambda_1^{2-\frac{n}{2}} \mu_2 > \lambda_2^{2-\frac{n}{2}} \mu_1$ respectively. Here $U^{(j)}$ is the unique¹ positive radial solution of $-\Delta u_j + \lambda_j u_j = \mu_j u_j^3$ in $W^{1,2}(\mathbb{R}^n)$, for $n = 1, 2, 3$ and $j = 1, 2$.

■

5. AN EXTENDED NLS-KdV SYSTEM WITH GENERAL POWER NONLINEARITIES

In this section we want to show that if one consider a more general system than (1), (3) with more general power nonlinearities, like the following

$$\begin{cases} i f_t + f_{xx} + \tau_1 |f|^{q-1} f + \beta f g &= 0 \\ g_t + g_{xxx} + \tau_2 |g|^{p-1} g_x + \frac{1}{2} \beta (|f|^2)_x &= 0, \end{cases} \quad (47)$$

where τ_1, τ_2, β are real constants, the one can prove the same results of the previous section with appropriate hypotheses. Looking for solutions of (47) in the form by (2) we find that for $\mu_1 = \tau_1$, $\mu_2 = \frac{\tau_2}{p}$, the real functions u, v solve the following system

$$\begin{cases} -u'' + \lambda_1 u &= \mu_1 |u|^{q-1} u + \beta u v \\ -v'' + \lambda_2 v &= \mu_2 |v|^{p-1} v + \frac{1}{2} \beta u^2, \end{cases} \quad (48)$$

where we consider $\lambda_j, \mu_j > 0$; $j = 1, 2$; $p, q \geq 2$. We take $\beta > 0$ in order to obtain positive solutions, although some results about the existence of bound states hold true too without the positivity of them.

Some of the results in Section 4 hold with minor changes. Notice that, since we look for positive solutions of (48), one could consider the term $(|g|^p)_x$ (as in previous sections where $p = 2$) instead of $|g|^{p-1} g_x$ in (47), and hence one would have $|g|^p$ instead of $|g|^{p-1} g$ in (48), obtaining the same existence of positive bound and ground states that we will prove here, in Theorems 13, 14. More general systems than (48) will be analyzed in a forthcoming paper.

¹See [15, 25] for this uniqueness result.

Note that (48) has a unique non-negative semi-trivial solution defined $\mathbf{v}_p = (0, V_p)$ with V_p the unique positive solution of $-v'' + \lambda_2 v = \mu_2 |v|^{p-1} v$ in H , which have the following explicit expression,

$$V_p(x) = \left[\frac{(p+1)\lambda_2}{2\mu_2 \cosh^2\left(\frac{p-1}{2}\sqrt{\lambda_2}x\right)} \right]^{\frac{1}{p-1}}. \quad (49)$$

Following similar notation as for (3), we denote the associated energy functional of (48) by

$$\Phi(\mathbf{u}) = J_1(u) + J_2(v) - \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v \, dx, \quad \mathbf{u} \in \mathbb{E}, \quad (50)$$

with

$$J_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{\mu_1}{q+1} \int_{\mathbb{R}} |u|^{q+1} dx, \quad J_2(v) = \frac{1}{2}\|v\|_2^2 - \frac{\mu_2}{p+1} \int_{\mathbb{R}} |v|^{p+1} dx; \quad u, v \in E.$$

Also, for $\Psi(\mathbf{u}) = (\nabla \Phi(\mathbf{u})|\mathbf{u})$, we define the corresponding Nehari manifold as

$$\mathcal{N} = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

Plainly,

$$(\nabla \Psi(\mathbf{u})|\mathbf{u}) = -\|\mathbf{u}\|^2 - \mu_1(q-2) \int_{\mathbb{R}} |u|^{q+1} dx - \mu_2(p-2) \int_{\mathbb{R}} |v|^{p+1} dx \quad \forall \mathbf{u} \in \mathcal{N},$$

thus \mathcal{N} is a smooth manifold locally near any point $\mathbf{u} \neq \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$. Moreover, $\Phi''(\mathbf{0}) = I_1''(0) + I_2''(0)$ is positive definite, so we infer that $\mathbf{0}$ is a strict minimum of Φ . As a consequence, $\mathbf{0}$ is an isolated point of the set $\{\Psi(\mathbf{u}) = 0\}$, proving that, on one hand \mathcal{N} is a smooth complete manifold of codimension 1, and on the other hand there exists a constant $\rho > 0$ so that

$$\|\mathbf{u}\|^2 > \rho, \quad \forall \mathbf{u} \in \mathcal{N}. \quad (51)$$

Then, as for (3) where $q = 3$, $p = 2$ one has that $\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\}$ is a critical point of Φ if and only if $\mathbf{u} \in \mathcal{N}$ is a critical point of Φ constrained on \mathcal{N} . Furthermore,

$$\Phi(\mathbf{u}) = \frac{1}{6}\|\mathbf{u}\|^2 + \left(\frac{1}{3} - \frac{1}{q+1}\right)\mu_1 \int_{\mathbb{R}} |u|^{q+1} dx + \left(\frac{1}{3} - \frac{1}{p+1}\right)\mu_2 \int_{\mathbb{R}} |v|^{p+1} dx \quad \forall \mathbf{u} \in \mathcal{N},$$

then clearly by (51) and the previous identity, Φ on \mathcal{N} is bounded bellow, for every $2 \leq p < \infty$, $2 \leq q < \infty$.

Proposition 12. *For Λ defined by (12):*

- (i) *if $\beta \leq \Lambda$, then \mathbf{v}_p is a strict local minimum of Φ constrained on \mathcal{N} ,*
- (ii) *for any $\beta > \Lambda$, then \mathbf{v}_p is a saddle point of Φ constrained on \mathcal{N} . Even more, $\inf_{\mathcal{N}} \Phi < \Phi(\mathbf{v}_2)$.*

The proof is a straightforward calculation of the one of Proposition 4. Furthermore, defining U_q as the unique positive solution of $-u'' + \lambda_1 u = \mu_1 |u|^{q-1} u$ in H (given by (49) substituting p by q), we have the following.

Theorem 13. *Assume that $2 \leq p < \infty$, $2 \leq q < \infty$.*

- (i) *If $\beta > \Lambda$, then System (48) has a positive even ground state $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$.*
- (ii) *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\tilde{\beta} > 0$, System (48) has an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0 = (U_q, V_p)$ as $\varepsilon \rightarrow 0$.*

Proof. We can adapt, with appropriate modifications, the ideas in the proof of Theorem 6, since the nonlinearity $|v|^{p+1}$ is even while in Theorem 6 the nonlinearity on v is v^3 (odd). That proves part (i).

Part (ii) follows by a little modification of the ideas of Theorem 9, by the same reason as above. ■

Concerning the existence of ground states for any $\beta > 0$ one can follow the proof of Theorem 8 that it holds true too with a restriction on the power exponent $q > 2p - 2$, which trivially holds for (3) where $q = 3$, $p = 2$. Thus, using that property, it is not difficult to show also the existence of positive bound states for $0 < \beta < \Lambda$ as in Theorem 10. We enunciate these results in the following.

Theorem 14. *Assume that $2 \leq p < \infty$, $2 \leq q < \infty$ and even more $q > 2p - 2$, then:*

- (i) *there exists $M > 0$ such that if $\lambda_2 > M$, System (48) has an even ground state $\tilde{\mathbf{u}} > \mathbf{0}$ for every $\beta > 0$,*
- (ii) *if $\lambda_2 > M$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.*

Remarks 15. (i) The restriction $q > 2p - 2$ appears when one tries to prove that

$$\Phi(t(V_p, V_p)) < \Phi(\mathbf{v}_p)$$

for $t(V_p, V_p) \in \mathcal{N}$. It does not seem to be optimal. Another test function different from $t(V_p, V_p)$ could circumvent this difficulty.

- (ii) When $p = 2$, $\mu_2 = p + 1$, in [18, Theorem 4.1] Dias et al. impose $\beta > 3$ to obtain even bound states. In our Theorem 13-(i), following the idea of Remark 5, it is easy to see that it holds for $\beta > 3 - a$ for some constant $a > 0$ when $\lambda_2 > \lambda_1$, obtaining positive even bound and ground states.
- (iii) Note that here, in Theorems 13, 14 we have $2 \leq p < \infty$, $2 \leq q < \infty$ obtaining positive even bound and ground states, in contrast with [18, Theorem 4.1] $2 < q < 5$, $p \in \{2, 3, 4\}$ and $\mu_2 = p + 1$ and in [3, Theorem 1.1] $2 \leq q < 5$, $2 \leq p < 5$ with p a rational number with odd denominator, where the authors obtained non-negative even bound states in the former and positive even bound states in the later.

■

6. FURTHER RESULTS

In this last section we show some results for explicit solutions. We point out some remarks and open problems. To finish, we study some extended systems with three or more equations.

6.1. Explicit solutions. In the particular case $0 < \beta < \frac{1}{6}$, $\lambda_2 = 4\lambda_1 + \frac{1}{12}\beta(1 - 6\beta)$ there exists a nontrivial explicit solution² $\mathbf{u}_\beta = (u_\beta, v_\beta)$ of (3) defined by

$$u_\beta(x) = \frac{\sqrt{2\lambda_1(1 - 6\beta)}}{\cosh(\sqrt{\lambda_1}x)}, \quad v_\beta(x) = \frac{12\lambda_1}{\cosh^2(\sqrt{\lambda_1}x)}.$$

Clearly one has that

$$\lim_{\beta \searrow 0} \mathbf{u}_\beta = \mathbf{u}_0 = (U_1, V_2), \quad \lim_{\beta \nearrow \frac{1}{6}} \mathbf{u}_\beta = \mathbf{v}_2 = (0, V_2), \quad (52)$$

²although the results in this subsection can be established for the more general system (48), we restrict ourselves to (3) for short.

where U_1, V_2 are defined by (29), (10) respectively. Then, the family $\{\mathbf{u}_\beta : 0 < \beta < \frac{1}{6}\}$ joins $\mathbf{u}_0 = (U_1, V_2)$ with \mathbf{v}_2 .

Remarks 16. (i) If we would have $\Phi(\mathbf{v}_2) \geq \Phi(\mathbf{u}_\beta)$ in the range $0 < \beta < \min\{\frac{1}{6}, \Lambda\}$, then we would be able to prove the existence of a positive even bound state \mathbf{u}^* with $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_\beta), \Phi(\mathbf{v}_2)\} = \Phi(\mathbf{v}_2)$, and in particular we would have a non-uniqueness of positive solutions result by a different way as in the previous sections. Unfortunately, if $0 < \beta < \frac{1}{6}$ then

$$\begin{aligned} \Phi(\mathbf{v}_2) &= \frac{1}{6} \|\mathbf{v}_2\|^2 = \frac{1}{12} \int_{\mathbb{R}} V_2^3 dx = \frac{9}{2} \lambda_2^3 \int_{\mathbb{R}} \frac{1}{\cosh^6(\frac{\sqrt{\lambda_2}}{2}x)} dx = \frac{24}{5} \lambda_2^{5/2} \\ &= \frac{24}{5} [4\lambda_1 + \frac{1}{12}\beta(1-6\beta)]^{5/2}, \end{aligned} \quad (53)$$

and

$$\begin{aligned} \Phi(\mathbf{u}_\beta) &= \frac{1}{6} \|\mathbf{u}_\beta\|^2 + \frac{1}{12} \int_{\mathbb{R}} u_\beta^4 dx = \frac{1}{6} (\|u_\beta\|_1^2 + \|v_\beta\|_2^2) + \frac{1}{12} \int_{\mathbb{R}} u_\beta^4 dx \\ &= \frac{1}{6} \left(\int_{\mathbb{R}} u_\beta^4 dx + \beta \int_{\mathbb{R}} u_\beta^2 v_\beta dx + \frac{1}{2} \left(\int_{\mathbb{R}} v_\beta^3 dx + \beta \int_{\mathbb{R}} u_\beta^2 v_\beta dx \right) \right) + \frac{1}{12} \int_{\mathbb{R}} u_\beta^4 dx \\ &= \frac{1}{4} \int_{\mathbb{R}} u_\beta^4 dx + \frac{\beta}{4} \int_{\mathbb{R}} u_\beta^2 v_\beta dx + \frac{1}{12} \int_{\mathbb{R}} v_\beta^3 dx \\ &= \lambda_1^2 (1-6\beta) \int_{\mathbb{R}} \frac{1}{\cosh^4(\sqrt{\lambda_1}x)} dx + 144\lambda_1^3 \int_{\mathbb{R}} \frac{1}{\cosh^6(\sqrt{\lambda_1}x)} dx \\ &= \frac{4}{3} \lambda_1^{3/2} (1-6\beta) + \frac{768}{5} \lambda_1^{5/2}. \end{aligned} \quad (54)$$

Comparing both energies, it is not difficult to show that $\Phi(\mathbf{v}_2) < \Phi(\mathbf{u}_\beta)$ for every $0 < \beta < \frac{1}{6}$.

- (ii) In the setting in which the explicit solutions \mathbf{u}_β of (3) exists, i.e., $0 < \beta < \frac{1}{6}$, $\lambda_2 = 4\lambda_1 + \frac{1}{12}\beta(1-6\beta)$, under hypotheses of Theorems 8 and 10, we have the multiplicity of positive solutions: the ground state $\tilde{\mathbf{u}}$ and the bound state \mathbf{u}^* . We conjecture that \mathbf{u}_β coincides with \mathbf{u}^* , and hence, the suggestive bifurcation diagram by Figure 1 holds not only for \mathbf{u}_β but also for \mathbf{u}^* .

■

6.2. Some systems with more than two equations. In this last subsection, we deal with some extended systems of (3) to more than two³ equations, but also consider (3) in a different dimensional case.

Note that System (1) has no sense in the dimensional case $n = 2, 3$, however, (3) makes sense to be extended to more dimensions. Moreover, the results of the previous Sections can be

³similar extensions of (48) to a more dimensional case, as (55) extends (3), can be considered (at least in the subcritical framework with $p, q < 2^*$ defined in Remark 17), proving similar results as Theorems 18, 20.

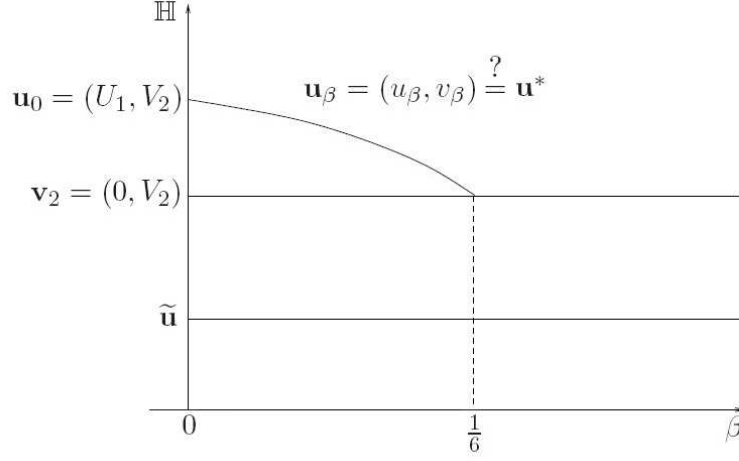


FIGURE 1. Suggestive bifurcation diagram of the family $\mathbf{u}_\beta = (u_\beta, v_\beta)$.

established in the dimensional case $n = 2, 3$ with minor changes for system

$$\begin{cases} -\Delta u + \lambda_1 u &= u^3 + \beta uv \\ -\Delta v + \lambda_2 v &= \frac{1}{2}v^2 + \frac{1}{2}\beta u^2, \end{cases} \quad (55)$$

working on the corresponding Sobolev Spaces $E = W^{1,2}(\mathbb{R}^n)$, $n = 2, 3$ and its radial subspace $H = E_r$. In particular, Theorems 6, 8, 9 and 10 hold, obtaining the corresponding positive bound and ground state solutions which are radially symmetric in this case.

Remarks 17. (i) For $n = 2, 3$ there is no lack of compactness, because there holds the compact embedding of the radial Sobolev Space H for all $2 < s < 2^*$ (see [28]), where $2^* = \infty$ if $n = 2$ and $2^* = \frac{2n}{n-2}$ for $n = 3$, which allow us to prove the Palais-Smale condition⁴ working on \mathbb{H} .

- (ii) Following some ideas by Ambrosetti and Colorado in [6], Liu and Zheng proved in [31] a partial result on existence of solutions to (55) in the dimensional case $n = 2, 3$. More precisely, in [31] the authors proved that the infimum of the functional associated to (55) on the corresponding Nehari manifold is achieved, but they do not proved that it is positive, and it was not shown that the infimum on the Nehari Manifold is a ground state, i.e., the least energy solution of the functional as we have proved here for $n = 1, 2, 3$. Also, in [31] was not investigated the existence of other bound states, as he have done in this manuscript and not only in the non-critical dimensions $n = 2, 3$ but also in the one dimensional case, $n = 1$.

■

System (55) can be seen as the stationary system of two coupled NLS-NLS equations when one looks for solitary wave solutions, and (u, v) are the corresponding standing wave solutions. It is well known that systems of NLS-NLS time-dependent equations have applications in some aspects of Optics, Hartree-Fock theory for Bose-Einstein condensates, among other physical phenomena;

⁴in a similar way as in [6, Lemma 3.2].

see for instance the earlier mathematical works [1, 4, 5, 6, 7, 10, 21, 26, 27, 32, 33, 34], the more recent list (far from complete) [11, 23, 30] and references therein.

By the above discussion, one can motivate, from the application point of view, the study of the following system of NLS-KdV-KdV equations,

$$\begin{cases} -\Delta u + \lambda_0 u &= u^3 + \beta_1 u v_1 + \beta_2 u v_2 \\ -\Delta v_1 + \lambda_1 v_1 &= \frac{1}{2} v_1^2 + \frac{1}{2} \beta_1 u^2 \\ -\Delta v_2 + \lambda_2 v_2 &= \frac{1}{2} v_2^2 + \frac{1}{2} \beta_2 u^2. \end{cases} \quad (56)$$

This system can also be seen as a perturbation of (55) if $n = 2, 3$ or a perturbation of (3) if $n = 1$, when $|\beta_1|$ or $|\beta_2|$ is small.

Now, we use the same notation as in previous sections with natural meaning, for example, $\mathbb{H} = H \times H \times H$, $\mathbb{E} = E \times E \times E$, $\mathbf{0} = (0, 0, 0)$,

$$\Phi(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 - \frac{1}{4} \int_{\mathbb{R}} u^4 dx - \frac{1}{6} \int_{\mathbb{R}} (v_1^3 + v_2^3) dx - \frac{1}{2} \int_{\mathbb{R}} u^2 (\beta_1 v_1 + \beta_2 v_2) dx \quad (57)$$

$$\mathcal{N} = \{\mathbf{u} \in \mathbb{E} \setminus \{\mathbf{0}\} : (\Phi'(\mathbf{u})|\mathbf{u}) = 0\}, \quad (58)$$

etc.

Let U^* , V_j^* be the unique positive radial solutions of $-\Delta u + \lambda u = u^3$, $-\Delta v + \lambda_j v = \frac{1}{2} v^2$ in E respectively, $j = 1, 2$; see [15, 25]. Then we have the following.

Theorem 18. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta_j = \varepsilon \tilde{\beta}_j > 0$, $j = 1, 2$, System (56) has a radial bound state \mathbf{u}_ε^* with $\mathbf{u}_\varepsilon^* \rightarrow \mathbf{u}_0^* = (U^*, V_1^*, V_2^*)$ as $\varepsilon \rightarrow 0$. Moreover, if $\beta_j > 0$ for $j = 1, 2$ then $\mathbf{u}_\varepsilon^* > \mathbf{0}$.*

The proof follows in a similar way as the proof of Theorem 9 with appropriate modifications, so that we omit it for short.

We also can prove the existence of a positive and radial ground state of (56) when the coupling parameters β_j , $j = 1, 2$ are sufficiently large. To do so, we define

$$\Lambda_j = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_0^2}{\int_{\mathbb{R}^n} V_j^* \varphi^2 dx} \quad j = 1, 2. \quad (59)$$

where $\|\cdot\|_0$ is the norm in E with λ_0 .

Remark 19. The unique non-negative semi-trivial solutions of (56) are given by $\mathbf{v}_1^* = (0, V_1^*, 0)$, $\mathbf{v}_2^* = (0, 0, V_2^*)$ and $\mathbf{v}_{12}^* = (0, V_1^*, V_2^*)$ with V_j^* for $j = 1, 2$ are defined before Theorem 57. ■

Concerning the ground states of (56), the first result is the following.

Theorem 20. *If $\beta_j > \Lambda_j$ for $j = 1, 2$, then (56) has a positive radial ground state $\tilde{\mathbf{u}}$.*

Proof. As in Proposition 4, using now that $\beta_j > \Lambda_j$, $j=1, 2$, one can show that both \mathbf{v}_1^* , \mathbf{v}_2^* are saddle points of the energy functional Φ (defined by (57)) constrained on the Nehari manifold \mathcal{N} (defined by (57)). Then

$$c = \inf_{\mathcal{N}} \Phi < \min\{\Phi(\mathbf{v}_1^*), \Phi(\mathbf{v}_2^*)\} < \Phi(\mathbf{v}_{12}^*) = \Phi(\mathbf{v}_1^*) + \Phi(\mathbf{v}_2^*). \quad (60)$$

By the Ekeland's variational principle, there exists a PS sequence $\{\mathbf{u}_k\}_{k \in \mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \rightarrow c \quad (61)$$

$$\nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \rightarrow 0. \quad (62)$$

If $n = 2, 3$, and $\{\mathbf{u}_k\}$ satisfies the PS condition; see Remark 17-(i). Thus, there exists a convergent subsequence (denoted equal for short) $\mathbf{u}_k \rightarrow \tilde{\mathbf{u}}$. Arguing in a similar way as in the proof of Theorem 6 we have that $\tilde{\mathbf{u}} \geq \mathbf{0}$ and moreover, by the Schwartz symmetrization properties, one can prove that

$$c = \Phi(\tilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E}, \Phi'(\mathbf{u}) = 0\}.$$

To prove the positivity of $\tilde{\mathbf{u}}$, if one supposes that the first component $u^* \equiv 0$, since the only non-negative solutions of (56) are the semi-trivial solutions defined in Remark 19, we obtain a contradiction with (60). Furthermore, if the second or third component vanish then $\tilde{\mathbf{u}}$ must be $\mathbf{0}$, and this is not possible because $\Phi|_{\mathcal{N}}$ is bounded below by a positive constant like in (8), then $\mathbf{0}$ is an isolated point of the set $\{\mathbf{u} \in \mathbb{H} : (\Phi'(\mathbf{u})|\mathbf{u}) = 0\}$, proving that \mathcal{N} is a complete manifold. Finally, the maximum principle gives that $\tilde{\mathbf{u}} > \mathbf{0}$.

In the one dimensional case, ($n = 1$), the lack of compactness discussed in Remarks 2-(ii) can be circumvent as in the proof of Theorem 6, proving the result. ■

Even more, one can show similar results to Theorems 8, 10 in the setting of (56). To do so, we first prove an auxiliary result in the following.

Proposition 21. *Assume that $\beta_j < \Lambda_j$, $j = 1, 2$ and moreover $\frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2} < 1$. Then \mathbf{v}_{12}^* is a strict local minimum of Φ constrained on \mathcal{N} .*

Proof. Notice that since \mathbf{v}_{12}^* is a critical point of Φ then $D^2\Phi_{\mathcal{N}}(\mathbf{v}_{12}^*)[\mathbf{h}]^2 = \Phi''(\mathbf{v}_{12}^*)[\mathbf{h}]^2$ for all $\mathbf{h} \in T_{\mathbf{v}_{12}^*}\mathcal{N}$. We denote

$$I_j(v) = \frac{1}{2}\|v\|_{\lambda_j}^2 - \frac{1}{6} \int_{\mathbb{R}^n} v^3 dx.$$

Then, using that V_j^* is a strict local minimum of I_j we have there exist two positive constants c_1, c_2 so that for $\mathbf{h} = (h_0, h_1, h_2) \in T_{\mathbf{v}_{12}^*}\mathcal{N}$,

$$\Phi''(\mathbf{v}_{12}^*)[\mathbf{h}]^2 \geq c_1\|h_1\|_1^2 + c_2\|h_2\|_2^2 + \|h_0\|^2 - \beta_1 \int_{\mathbb{R}^n} h_0^2 V_1^* dx - \beta_2 \int_{\mathbb{R}^n} h_0^2 V_2^* dx. \quad (63)$$

From (63) and $\frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2} < 1$ we infer that

$$\Phi''(\mathbf{v}_{12}^*)[\mathbf{h}]^2 \geq c_1\|h_1\|_1^2 + c_2\|h_2\|_2^2 + \left(1 - \frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2}\right) \|h_0\|^2,$$

which proves the result. ■

Theorem 22. *Assume that $\beta_1, \beta_2 > 0$, for λ_1, λ_2 large enough:*

- (i) *there exists a radial ground state $\tilde{\mathbf{u}} > \mathbf{0}$,*
- (ii) *if additionally $\frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2} < 1$, there exists a radial bound state $\mathbf{u}^* > \mathbf{0}$. Furthermore,*

$$\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_{12}^*) = \Phi(\mathbf{v}_1^*) + \Phi(\mathbf{v}_2^*).$$

Proof. The proof of (i) follows in a similar way as the one of Theorem 8 with appropriate changes, then we omit details for short. With respect to (ii), using Proposition 21, one can modify appropriately the arguments of the proof of Theorem 10 to obtain the result, so once again we can omit the complete details. ■

Remark 23. It is easy to extend these results to systems with any number of equations $N > 3$ as the following

$$\begin{cases} -\Delta u + \lambda_0 u &= u^3 + \sum_{k=1}^{N-1} \beta_k u v_k \\ -\Delta v_j + \lambda_j v_j &= \frac{1}{2} v_j^2 + \frac{1}{2} \beta_j u^2; \quad j = 1, \dots, N-1. \end{cases} \quad (64)$$

For example, the existence of a positive radial ground state of (64) provided

$$\beta_k > \Lambda_k = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_0^2}{\int_{\mathbb{R}} V_k^* \varphi^2 dx} \quad k = 1, \dots, N-1,$$

where V_k^* is the unique positive radial solution of $\Delta v + \lambda_k v = \frac{1}{2} v^2$ in E , $k = 1, \dots, N-1$. ■

Another natural extension of (3) to more than two equations different from (56) is the following system of NLS-NLS-KdV equations,

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= u_1^3 + \beta_{12} u_1 u_2^2 + \beta_{13} u_1 v \\ -\Delta u_2 + \lambda_2 u_2 &= u_2^3 + \frac{1}{2} \beta_{12} u_1^2 u_2 + \beta_{23} u_2 v \\ -\Delta v + \lambda v &= \frac{1}{2} v^2 + \frac{1}{2} \beta_{13} u_1^2 + \frac{1}{2} \beta_{23} u_2^2. \end{cases} \quad (65)$$

Here we obtain a bifurcation result for this system in a similar way as in Theorems 9, 18, that we enunciate in the following.

Theorem 24. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta_{jk} = \varepsilon \tilde{\beta}_{jk}$, $k = 1, 2$, $j = 2, 3$, $k \neq j$, System (65) has a radial bound state \mathbf{u}_ε^* with $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}_0^* = (U_1, U_2, V)$ as $\varepsilon \rightarrow 0$. Moreover, if all $\beta_{kj} > 0$ then $\mathbf{u}_\varepsilon^* > \mathbf{0}$.*

Where U_j is the unique positive radial solution of $-\Delta u + \lambda_j u = u^3$ in E ; $j = 1, 2$; and V the corresponding positive radial solution to $-\Delta v + \lambda v = \frac{1}{2} v^2$ in E .

Note that the non-negative radial semi-trivial solution $(0, 0, V)$ is a strict local minimum of the associated energy functional constrained on the corresponding Nehari manifold provided

$$\beta_{j3} < \Lambda_j = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_{\lambda_j}^2}{\int_{\mathbb{R}^n} V \varphi^2 dx} \quad j = 1, 2.$$

While if either $\beta_{13} > \Lambda_1$ or $\beta_{23} > \Lambda_2$ then $(0, 0, V)$ is a saddle critical point of Φ on \mathcal{N} .

There also exist semi-trivial solutions coming from the solutions studied in Section 4, with the first or the second component $\equiv 0$. This fact makes different the analysis of (65) with respect to the previous studied systems (56) and (64).

Finally, one could study more general extended systems of (56), (65) with $N = m + \ell$; m -NLS and ℓ -KdV coupled equations with $m, \ell \geq 2$ in the one dimensional case, or N -NLS equations if $n = 2, 3$. A careful analysis of this kind of systems including (65) will be done in a forthcoming paper.

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